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Domination edge connectivity of graphs

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ABSTRACT

Domination and connectivity are two independent subjects of graph theory which have many applications in computer and information sciences. To bring these two terms together, we first define a novel conditional connectivity measure: k- domination edge connectivity. Let $G = (V, E)$ be a connected graph and S is the edge set of G. If $G - S$ is disconnected and every disconnected component has domination number equals k, then minimum cardinality of the set S is called k-domination edge connectivity number of G and denoted as $\lambda^{\gamma(k)}(G)$. In this study we compute 2-domination edge connectivity of paths, cycles and complete graphs.

Keywords: Soft computing, connectivity, domination, conditional connectivity, domination connectivity, k-domination edge connectivity

INTRODUCTION

Domination notion in computer science has important application in VLSI design problems. Also, domination is being used to solve security problems for interconnection networks.

Approximately forty different domination parameters have been defined in graph theory so far [1,2]. Connectivity is indispensable tool for measuring fault tolerance capacity of networks [3]. Conditional connectivity is a special type connectivity was invented by Harary in 1983 [4]. Harary's approach is related to assume that every disconnected component subnetwork has desired some special desired properties. After this seminal study, a couple of novel conditional connectivity measures have been defined for understanding fault tolerance capacity of multi-processor systems [5-13].

We notice that any conditional connectivity measurement based to both domination and connectivity notions are not exist in the related literature. Because of fill in this gap, we decide to define a novel conditional connectivity invariant of graph theory: Domination (edge) connectivity.

PRELIMINARIES

Necessary definitions are given in order to prepare the reader for calculations of the 2-domination edge connectivity.

Let $G=(V,E)$ a connected graph where V is the vertex set and E is the edge set. The degree of any vertex(atom, node) of G is the number of edges(bond, link) incident to this vertex and denoted as, deg v for the vertex v of V. If all the degrees of the vertices of G equal r, then G is called r-regular graph.2-regular connected graphs with n-edge and n-vertex are called cycles and denoted as C_n. If an edge deleted from the cycle C_n then the path graph P_n is acquired. If an n-vertex graph, every vertex is adjacent to the other all vertices then this graph is called complete graph and denoted as K_n.

Edge connectivity of a connected graph G, $\lambda(G)$, is the minimum number of edges whose deletion make the graph G disconnected. And now we firstly give the definition of a novel conditional connectivity measure in graph theory literature; k-domination edge connectivity

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such as: Let $G = (V, E)$ be a connected graph and S is the edge set of G. If $G - S$ is disconnected and every component has domination number is equal to k, then minimum cardinality of the set S is called k-domination edge connectivity number of G and denoted as $\lambda^{\gamma(k)}(G)$.

And now, we begin to compute the 2-domination edge connectivity for paths, cycles, and complete graphs. We use combinatorial computing techniques method in our computations.

RESULTS

Proposition 1. For $n \geq 8$, $\lambda^{\gamma(2)}(P_n) = \left\lfloor \frac{n-1}{6} \right\rfloor$.

Proof. We know from the definition of 2-domination edge connectivity that every disconnected component of P_n has domination number two. Therefore, all disconnected components must be following paths P_4, P_5 or P_6, since they are only paths have domination number two. Each path graph can be disconnected into by using three, two or one of these suitable graphs P_4, P_5 or P_6 according to n. The following table shows possible disconnected components for some n values.

Table 1. Possible disconnected decompositions of P_n with respect to P_4, P_5 and P_6 for $n \geq 8$.

P_n	Disconnected decompositions of P_n	P_n	Disconnected decompositions of P_n
P_8	P_4, P_4	P_{14}	P_4, P_5, P_5
P_9	P_4, P_5	P_{15}	P_5, P_5, P_5
P_{10}	P_5, P_5	P_{16}	P_5, P_5, P_6
P_{11}	P_5, P_6	P_{17}	P_5, P_6, P_6
P_{12}	P_6, P_6	P_{18}	P_6, P_6, P_6
P_{13}	P_4, P_4, P_5	P_{19}	P_4, P_5, P_5, P_5
P_{20}	P_5, P_5, P_5, P_5	P_{35}	$P_5, P_6, P_6, P_6, P_6, P_6$

P_{21}	P_5, P_5, P_5, P_6	P_{36}	$P_6, P_6, P_6, P_6, P_6, P_6$
P_{22}	P_5, P_5, P_6, P_6	P_{37}	$P_5, P_5, P_5, P_5, P_5, P_6, P_6$
P_{23}	P_5, P_6, P_6, P_6	P_{38}	$P_5, P_5, P_5, P_5, P_6, P_6, P_6$
P_{24}	P_6, P_6, P_6, P_6	P_{39}	$P_5, P_5, P_5, P_6, P_6, P_6, P_6$
P_{25}	P_5, P_5, P_5, P_5, P_5	P_{40}	$P_5, P_5, P_6, P_6, P_6, P_6, P_6$
P_{26}	P_5, P_5, P_5, P_5, P_6	P_{41}	$P_5, P_6, P_6, P_6, P_6, P_6, P_6$
P_{27}	P_5, P_5, P_5, P_6, P_6	P_{42}	$P_6, P_6, P_6, P_6, P_6, P_6, P_6$
P_{28}	P_5, P_5, P_6, P_6, P_6	P_{43}	$P_5, P_5, P_5, P_5, P_5, P_6, P_6, P_6$
P_{29}	P_5, P_6, P_6, P_6, P_6	P_{44}	$P_5, P_5, P_5, P_5, P_6, P_6, P_6, P_6$
P_{30}	P_6, P_6, P_6, P_6, P_6	P_{45}	$P_5, P_5, P_5, P_6, P_6, P_6, P_6, P_6$
P_{31}	$P_5, P_5, P_5, P_5, P_5, P_6$	P_{46}	$P_5, P_5, P_6, P_6, P_6, P_6, P_6, P_6$
P_{32}	$P_5, P_5, P_5, P_5, P_6, P_6$	P_{47}	$P_5, P_6, P_6, P_6, P_6, P_6, P_6, P_6$
P_{33}	$P_5, P_5, P_5, P_6, P_6, P_6$	P_{48}	$P_6, P_6, P_6, P_6, P_6, P_6, P_6, P_6$
P_{34}	$P_5, P_5, P_6, P_6, P_6, P_6$	P_{49}	$P_5, P_5, P_5, P_5, P_5, P_6, P_6, P_6, P_6$

Notice from the Table 1 that every P_n for $n \geq 26$ can be disconnected into both P_5 and P_6 paths for n is not congruent to zero with modulo six or only P_6 paths for n is congruent to zero with modulo six. Based on the pigeonhole principle, the key is to use as many P_6 paths as possible. Thus, the least number of edges will be deleted. There are six cases in this proof.

Case 1: $n \equiv 0\ (mod\ 6)$. All disconnected components must be P_6 in this case. Therefore, it is enough to delete $\frac{n}{6} - 1 = \left\lfloor \frac{n-1}{6} \right\rfloor$ suitable edges from P_n.

Case 2: $n \equiv 1\ (mod\ 6)$. Let $n = 6k + 1$. In this case we must disconnected P_n into five number of P_5 and $k - 4$ number of P_6 since $n = 5 \times 5 + (k - 4) \times 6$. Therefore, the number of disconnected components is $k+1$. And we need to delete at least k number of suitable edges from P_n. Thus, $k=\left\lfloor\frac{n-1}{6}\right\rfloor$.

Case 3: $n \equiv 2\ (mod\ 6)$. Let $n=6k+2$. In this case we must disconnected P_n into four number of P_5 and $k-3$ number of P_6 since $n=4\times 5+(k-3)\times 6$. Therefore, the number of disconnected components is $k+1$. And we need to delete at least k number of suitable edges from P_n. Thus, $k=\left\lfloor\frac{n-1}{6}\right\rfloor$.

Case 4: $n \equiv 3\ (mod\ 6)$. Let $n=6k+3$. In this case we must disconnected P_n into three number of P_5 and $k-2$ number of P_6 since $n=3\times 5+(k-2)\times 6$. Therefore, the number of disconnected components is $k+1$. And we need to delete at least k number of suitable edges from P_n. Thus, $k=\left\lfloor\frac{n-1}{6}\right\rfloor$.

Case 5: $n \equiv 4\ (mod\ 6)$. Let $n=6k+4$. In this case we must disconnected P_n into two number of P_5 and $k-1$ number of P_6 since $n=2\times 5+(k-1)\times 6$. Therefore, the number of disconnected components is $k+1$. And we need to delete at least k number of suitable edges from P_n. Thus, $k=\left\lfloor\frac{n-1}{6}\right\rfloor$.

Case 6: $n \equiv 5\ (mod\ 6)$. Let $n=6k+5$. In this case we must disconnected P_n into one number of P_5 and k number of P_6 since $n=1\times 5+k\times 6$. Therefore, the number of disconnected components is $k+1$. And we need to delete at least k number of suitable edges from P_n. Thus, $k=\left\lfloor\frac{n-1}{6}\right\rfloor$.

Corollary 2. For $n \geq 8$, $\lambda^{\gamma(2)}(C_n)=\left\lceil\frac{n}{6}\right\rceil$.

Proof. Since we obtain P_n only one edge deletion from C_n, it is enough to add one to the result of the Proposition 1. Therefore, $\lambda^{\gamma(2)}(C_n)=\left\lfloor\frac{n-1}{6}\right\rfloor+1=\left\lceil\frac{n}{6}\right\rceil$.

Following Table 2 shows computer aided calculations of 2-domination connectivity number of some complete graphs.

Table 2. 2-domination connectivity number of some complete graphs.

K_n	$\lambda^{\gamma(2)}(K_n)$	K_n	$\lambda^{\gamma(2)}(K_n)$
K_8	20	K_{16}	56
K_9	25	K_{17}	61
K_{10}	29	K_{18}	65
K_{11}	34	K_{19}	70
K_{12}	38	K_{20}	74
K_{13}	43	K_{21}	79
K_{14}	47	K_{22}	83
K_{15}	52	K_{23}	88

Before calculating 2-domination connectivity in complete graphs, we give two lemmas that we will use with proof of Proposition 5.

Lemma 3. Let G be $(n-2)$-regular connected graph with n vertex for $n \geq 4$ and even, then $\gamma(G) = 2$.

Proof. Without loss of generality, we choose two non-neighbour vertices u and v. Since u has $n-2$ neighbouring vertices, u dominates all of the vertices of G except the non-neighbouring vertex v. Therefore, the set $\{u, v\}$ dominates every vertex of G. Thus, $\gamma(G) = 2$.

Lemma 4. Let G be connected graph with $(n-2)$ degree of $(n-1)$-vertex and $(n-3)$ degree of one vertex for $n \geq 5$ and odd, then $\gamma(G) = 2$.

Proof. Without loss of generality, we choose two non-neighbour vertices u and v. Let u has $n-2$ neighbouring vertices and, v is the only vertex has $n-3$ degree. Since u has $n-2$ neighbouring vertices, u dominates all of the vertices of G except the non-neighbouring vertex v. Therefore, the set $\{u, v\}$ dominates every vertex of G. Thus, $\gamma(G) = 2$.

We can now examine 2-domination connectivity in complete graphs.

Proposition 5. For $n \geq 8$, $\lambda^{\gamma(2)}(K_n) = \begin{cases} \frac{9n-32}{2} & \text{for } n \text{ is even,} \\ \frac{9n-31}{2} & \text{for } n \text{ is odd.} \end{cases}$

Proof. There are two cases.

Case1: Let n is even. According to the definition of 2-domination connectivity, we must disconnect K_n into C_4 and $(n-6)$-regular graph with $n-4$ vertex graph named as S. All remaining disconnection procedure which is made by using 2-domination connectivity, requires deleting more edges. We know that $\gamma(S) = 2$ from Lemma 3 and $\gamma(C_4) = 2$. Complete graph K_n has $\frac{n(n-1)}{2} = \frac{n^2-n}{2}$ edges. Notice that S has $4 + \frac{(n-6)(n-4)}{2} = \frac{n^2-10n+32}{2}$ edges. The total number of deleted suitable edges after implying definition of 2-domination connectivity to K_n is $\frac{n^2-n}{2} - \frac{n^2-10n+32}{2} = \frac{9n-32}{2}$.

Case 2: Let n is odd. According to the definition of 2-domination connectivity, we must disconnect K_n into C_4 and $(n-6)$ degree of $(n-5)$-vertex, $(n-7)$ degree of one vertex graph named as S. All remaining disconnection procedure which is made by using 2-domination connectivity, requires deleting more edges. We know that $\gamma(S) = 2$ from Lemma 4 and $\gamma(C_4) = 2$. Complete graph K_n has $\frac{n(n-1)}{2} = \frac{n^2-n}{2}$ edges. Notice that S has $4 + \frac{(n-6)(n-5)+(n-7)}{2} = \frac{n^2-10n+31}{2}$ edges. The total number of deleted suitable edges after implying definition of 2-domination connectivity to K_n is $\frac{n^2-n}{2}-\frac{n^2-10n+31}{2}=\frac{9n-31}{2}$. Thus, the proof is completed.

We can give following corollary which joins two cases of Proposition 5, by using properties of ceiling function.

Corollary 6. $\lambda^{\gamma(2)}(K_n)=4n+\left\lceil\frac{n}{2}\right\rceil-16$.

CONCLUSION

In this study, a novel field of study named as k-domination edge connectivity is emerged by combining the concepts of domination and connectivity terms which are two independent subjects of graph theory. We compute 2-domination edge connectivity of paths, cycles, and complete graphs. This study has a potential to stimulate further the following researches:

- Computing 2-domination edge connectivity in graph operations,
- Reckoning k-domination edge connectivity for paths, cycles, complete graphs, alternating group networks and graph operations for different values of k,
- Calculating k-domination for well-known interconnection networks,
- Defining k-domination vertex connectivity and analysing mathematical properties,
- Computing k-domination vertex connectivity for graph classes,
- Reckoning k-domination vertex connectivity for interconnection networks.
- Defining and analysing other type of domination connectivity such as: total domination connectivity, restrained domination connectivity, rainbow domination connectivity, stratified domination connectivity, Roman domination connectivity etc.

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